Wave propagation in a uniaxial media

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Outline:

Polarization of light waves

Electromagnetic wave propagation in uniaxial Media

Polarization of light waves

- Oscillation of the electric field strongly affects the light propagation in a media, especially, anisotropic material.

- Observation of the polarization of the plane wave

- Polarization of the plane wave
  - Polarization : the direction of the ***E(r, t)***
  - ***E(r, t)*** : oscillates with sinusoidal time-harmonic ftn.
  - If light propagates to *z* direction, the components of the electric field should be *xy* plane

$$\vec{E}(z,t) = \mathrm{Re}(\vec{A}e^{i(\omega t - kz)})$$

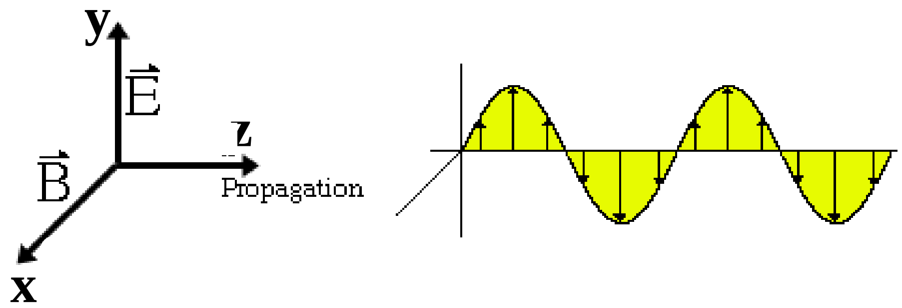
Complex representation

$$\vec{A} = Ax\, e^{i\delta x}\,\overrightarrow{ax} + Ay\, e^{i\delta y}\,\overrightarrow{ay}$$

– The curve of which the end point of the electric field E describes the time-evolution locus and has 2 x-y components

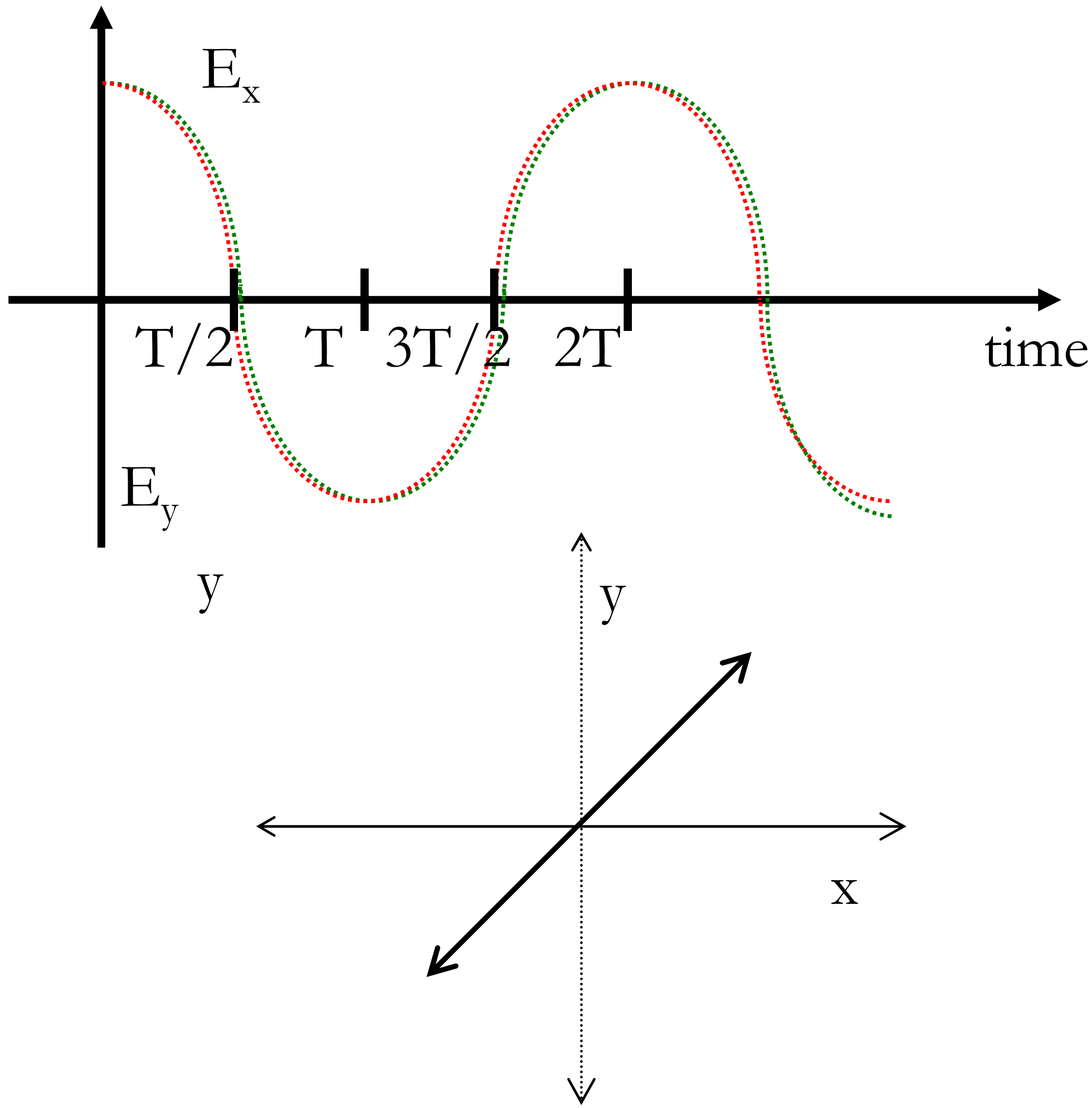
$$E_x = A_x \cos(\omega t - kz - \delta_x)$$

$$E_y = A_y \cos(\omega t - kz - \delta_y)$$

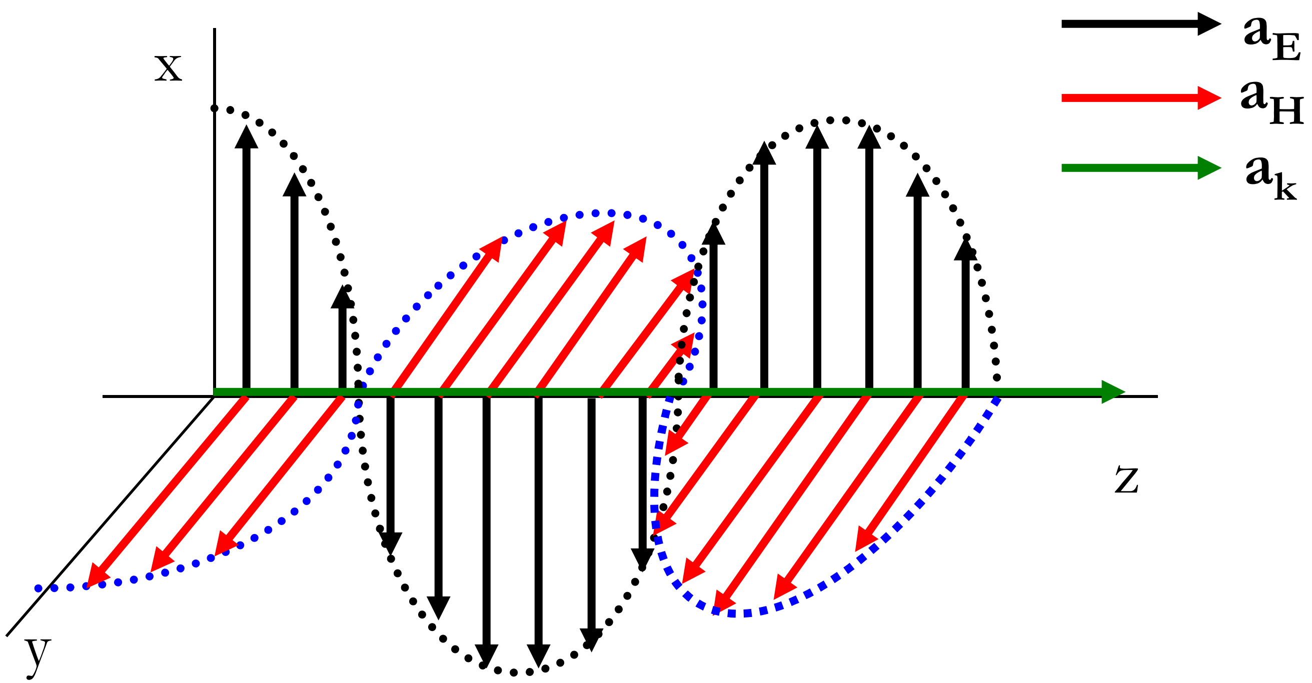


- Revolution direction :

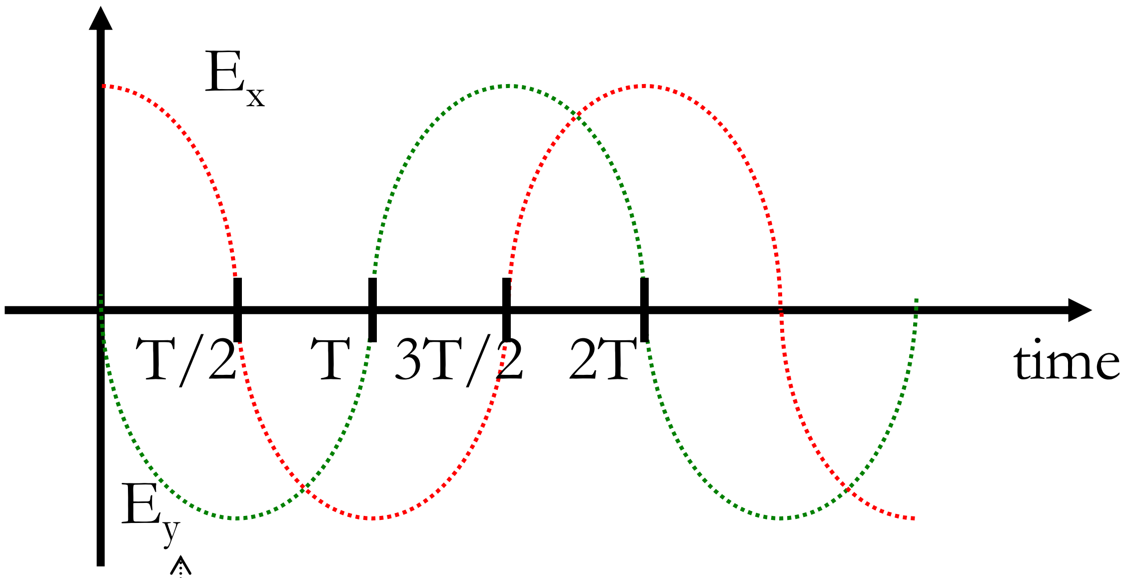
1) in case of $A_x = A_y = 1$ $\delta x = 0, \delta y = 0$ ($z=0$)

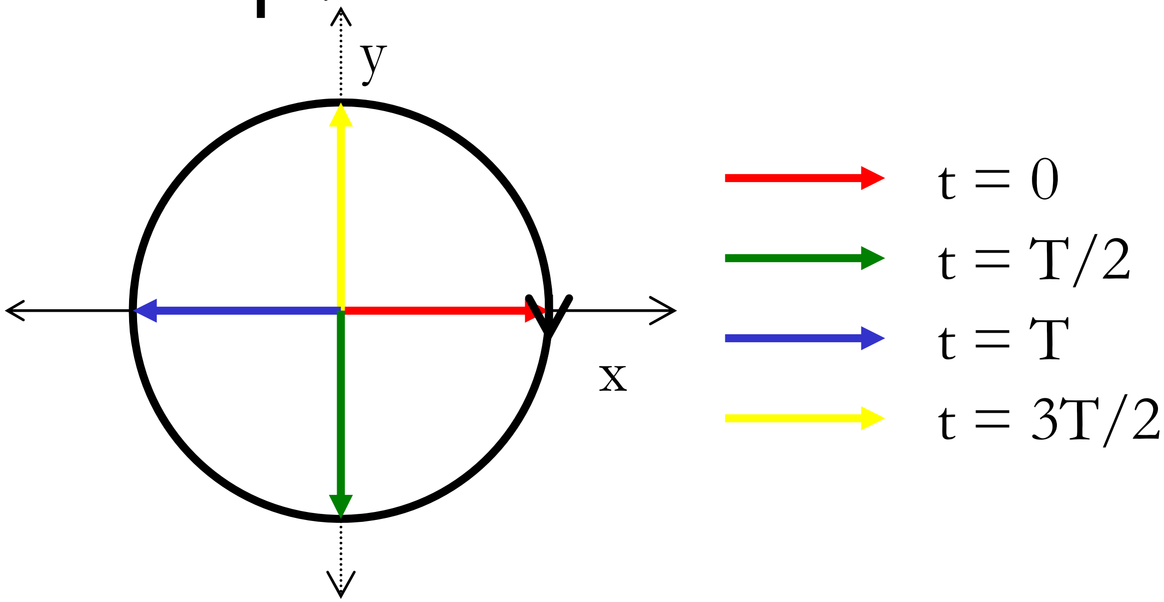


Spatial domain (t=0)



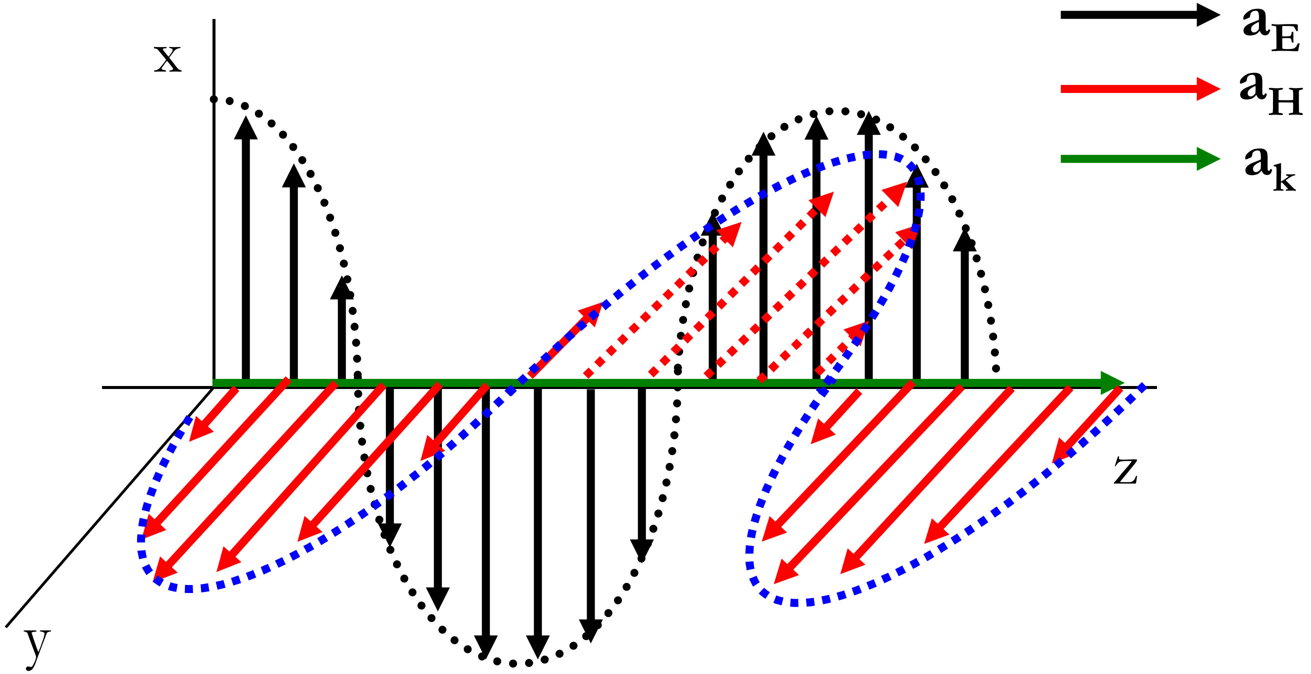
2) in case of $A_x = A_y = 1$ $\delta x = 0, \delta y = \pi/2$ $(z = 0)$



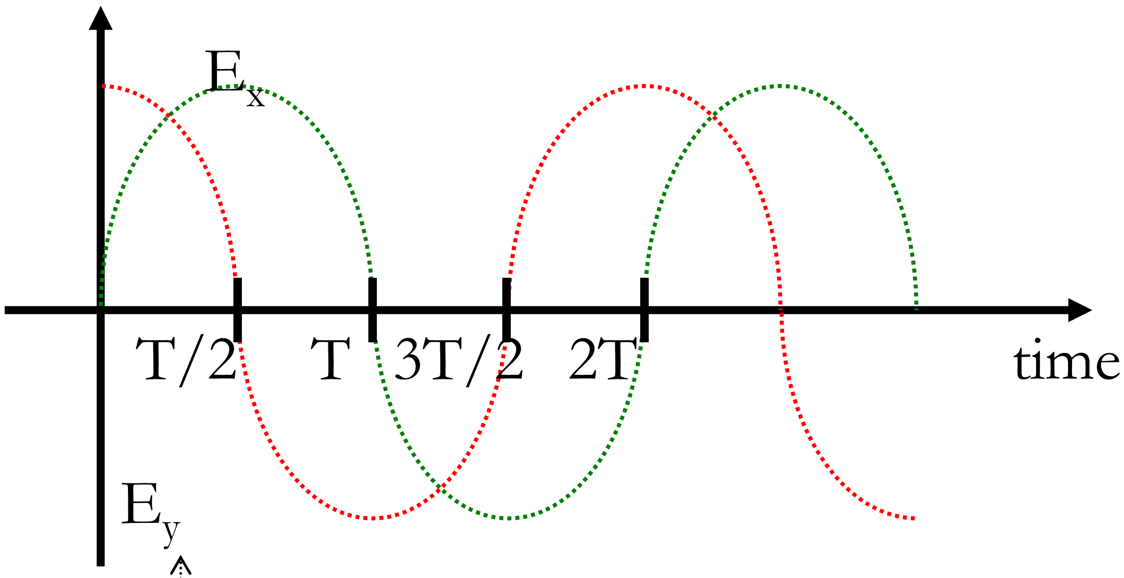


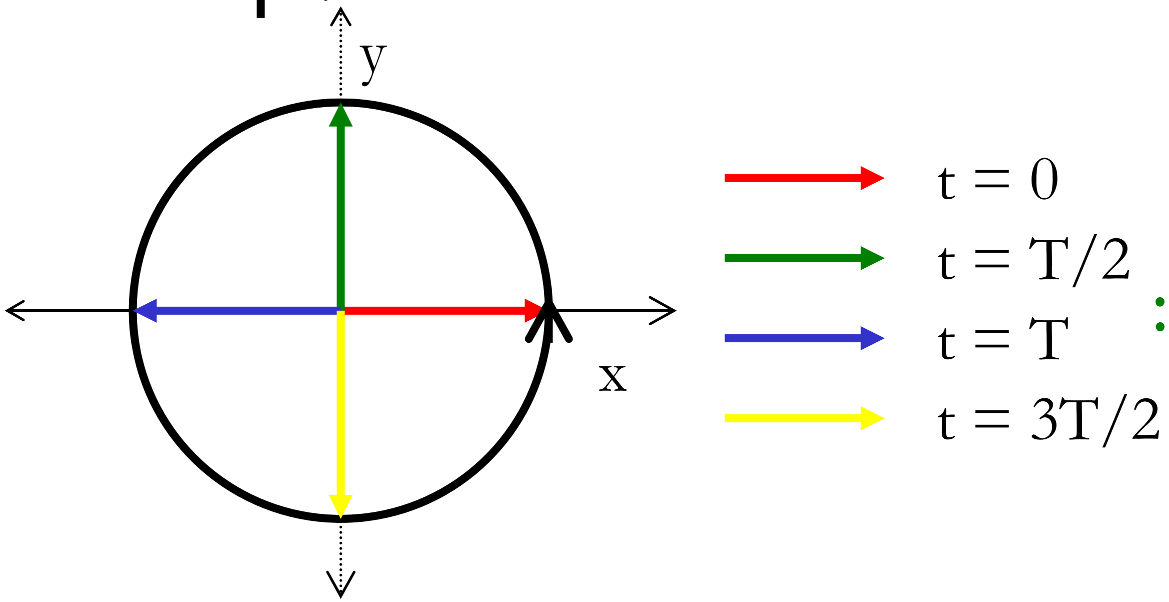
: clockwise

Spatial domain (t=0)



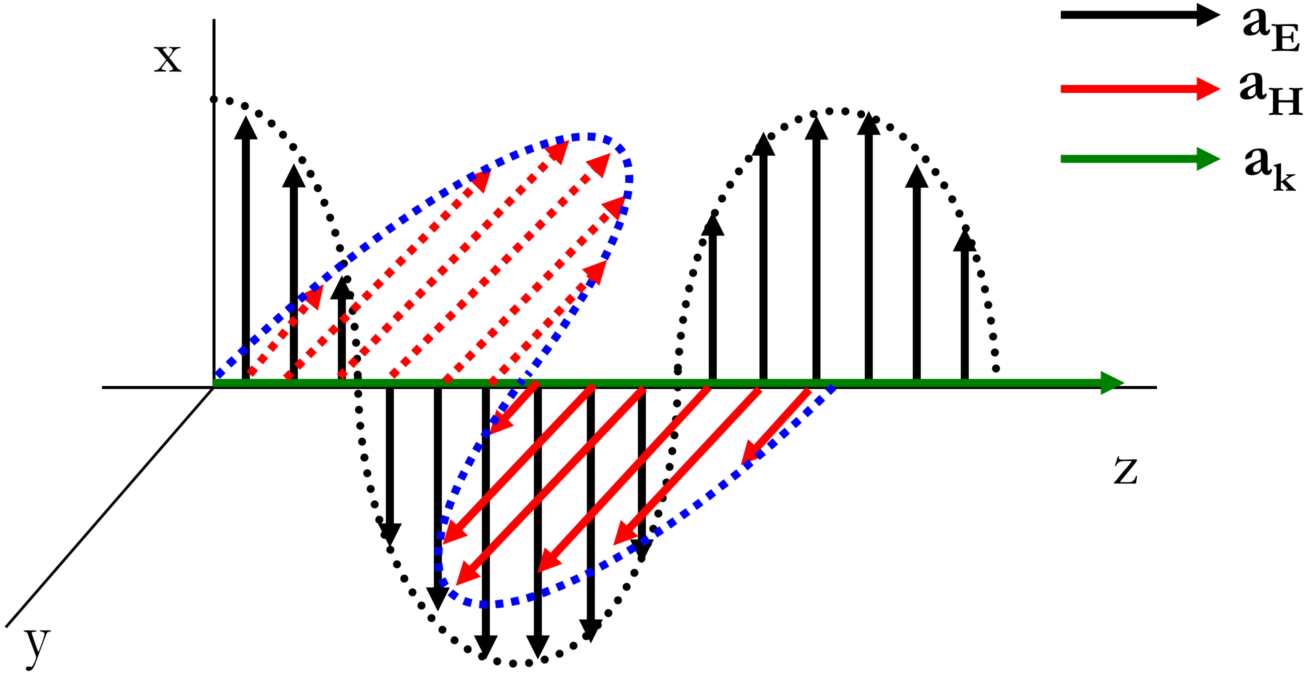
3) in case of Ax = Ay = 1 $\delta x = 0, \delta y = -\pi/2$





: counterclockwise

Spatial domain (t=0)

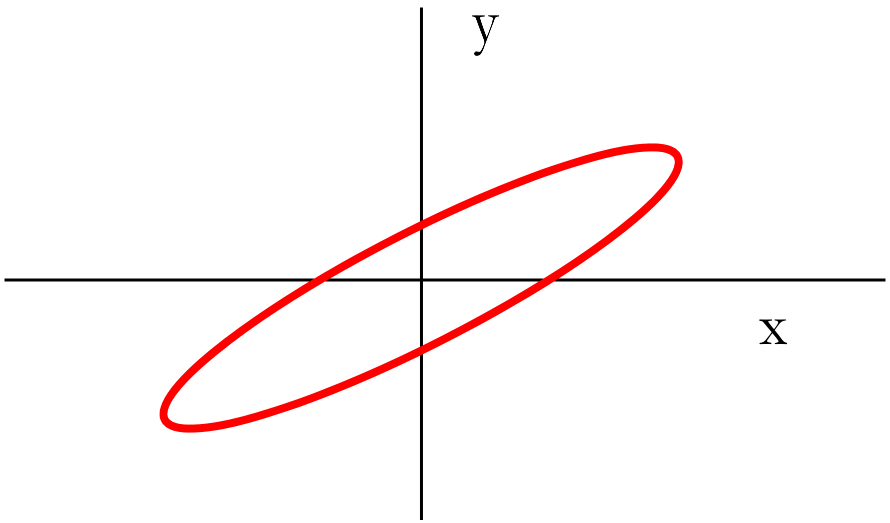


- Typical polarization ellipse

From electric field components E_x, E_y

$$\left(\frac{E_x}{A_x}\right)^2 + \left(\frac{E_y}{A_y}\right)^2 - 2\frac{\cos\delta}{A_x A_y} E_x E_y = \sin^2\delta$$

$$\delta = \delta_y - \delta_x$$



- Rotation to the Principal axis

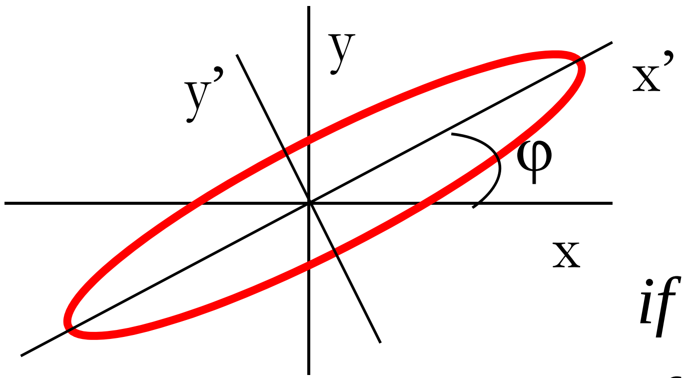
$$\left(\frac{E_{x'}}{a}\right)^2 + \left(\frac{E_{y'}}{b}\right)^2 = 1$$

Diagonalization → x' y' (principal axis)

$$a = \sqrt{A_x^{\ 2}\cos^2\varphi + A_y^{\ 2}\sin^2\varphi + 2A_xA_y\cos\delta\cos\varphi\sin\varphi}$$

$$b = \sqrt{A_x^{\ 2}\sin^2\varphi + A_y^{\ 2}\cos^2\varphi - 2A_xA_y\cos\delta\cos\varphi\sin\varphi}$$

$$\varphi = \tan^{-1}\left(\frac{2A_xA_y}{A_x^{\ 2} - A_y^{\ 2}}\cos\delta\right)/2$$

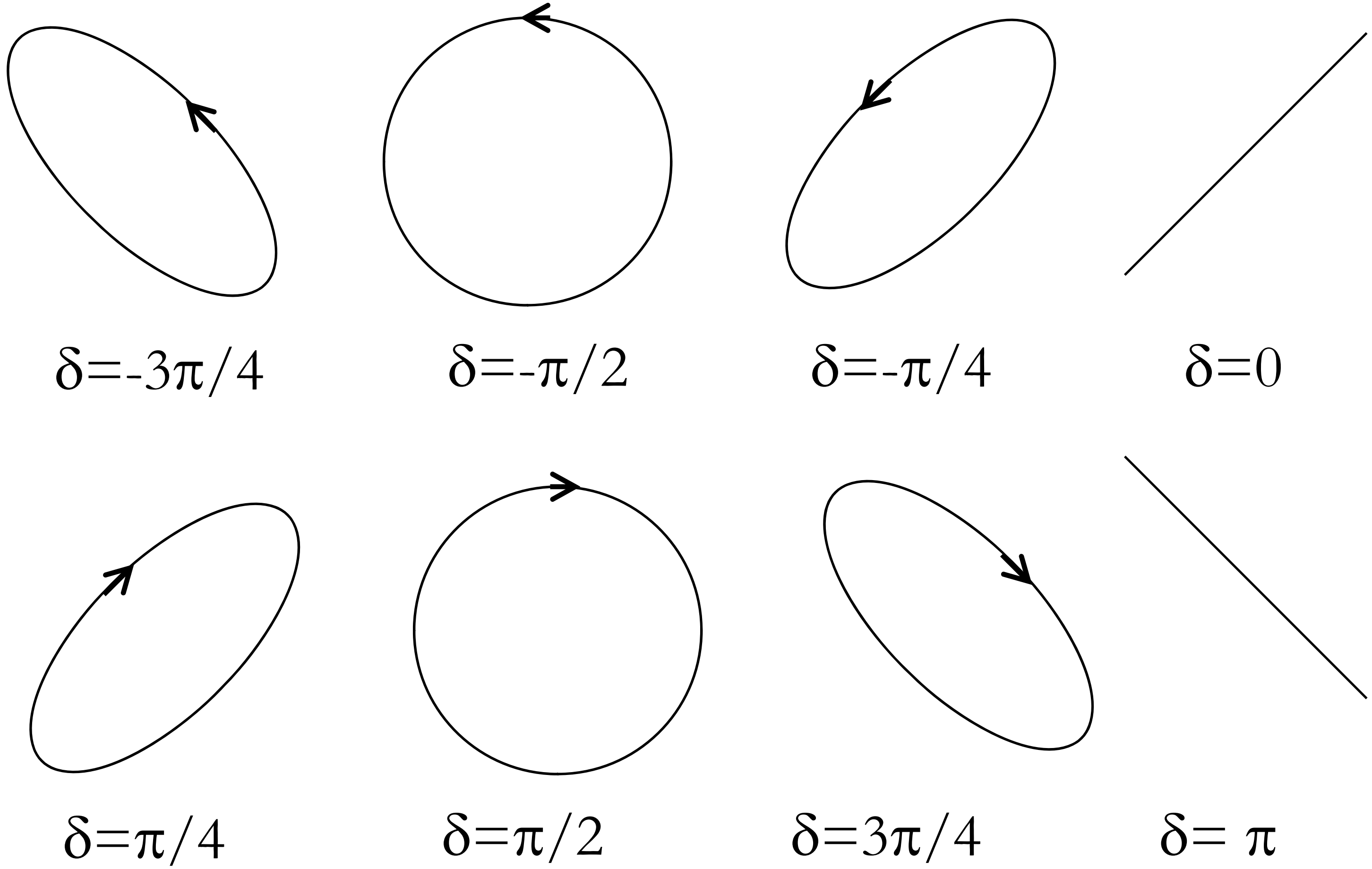


$$if\ \ \sin\delta > 0\ \ then\ \ colckwise$$

$$if\ \ \sin\delta < 0\ \ then\ \ countercolckwise$$

Ex1) Polarization ellipses : $E_x = \cos(\omega t - kz)$

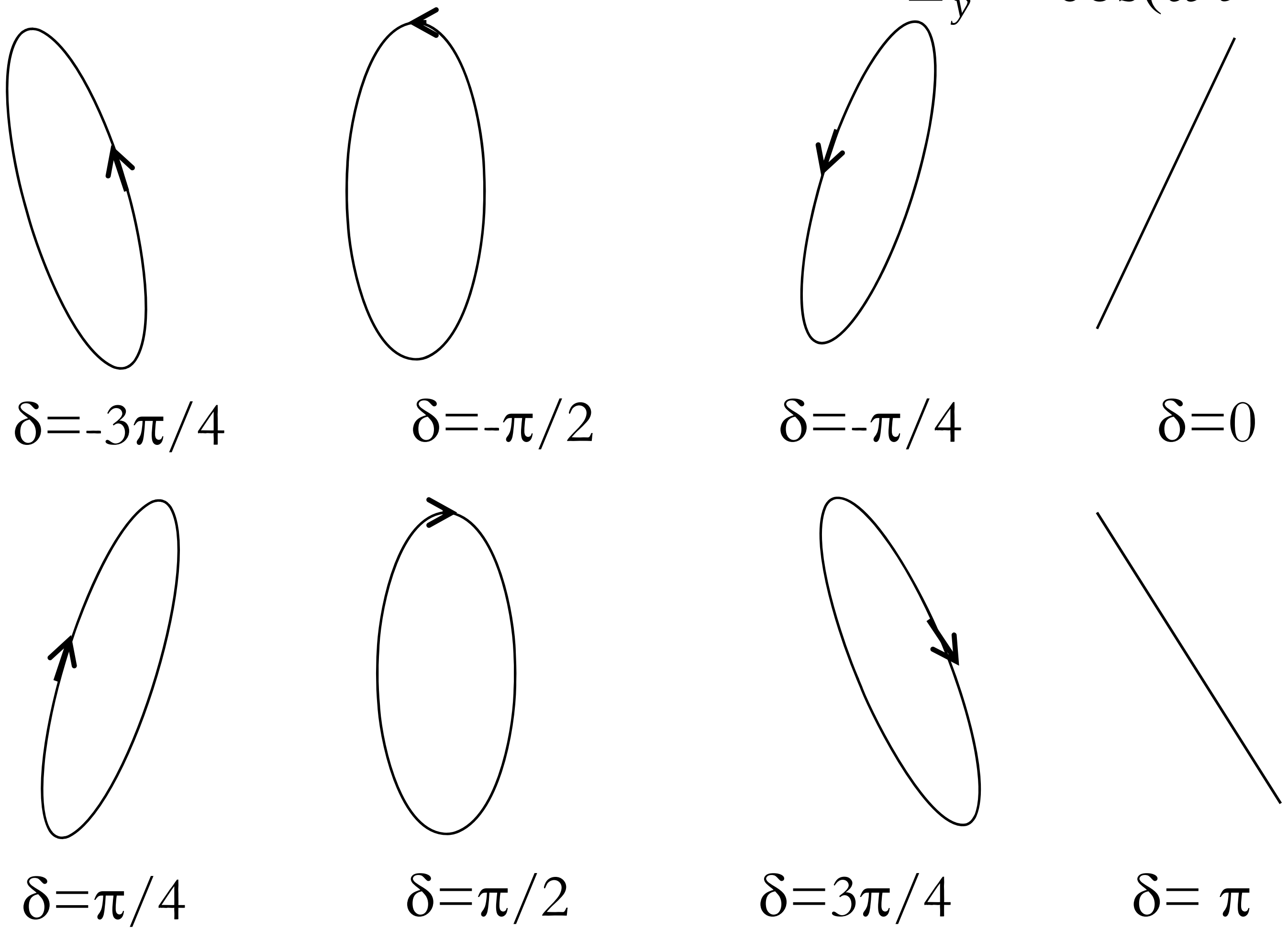
$$E_y = \cos(\omega t - kz + \delta)$$



Ex2) Polarization ellipses :

$$E_x = \frac{1}{2}\cos(\omega t - kz)$$

$$E_y = \cos(\omega t - kz + \delta)$$



Electromagnetic wave propagation in Anisotropic Media

- Anisotropic materials : calcite, quartz, LC, etc
- They show very particular phenomenon such as double refraction, optical rotation, polarization effect, etc
- Analytic description will de shown at here

• Dielectric tensor

Flux density ***D***

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

Induced polarization ***P***

Isotropic material : ***P*** // ***E***

Anisotropic material : ***P*** not // ***E***

$$\vec{P} = \varepsilon_0 \chi \vec{E}$$

$$\begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix} = \varepsilon_0 \begin{pmatrix} \chi_{11} & \chi_{12} & \chi_{13} \\ \chi_{21} & \chi_{22} & \chi_{23} \\ \chi_{31} & \chi_{32} & \chi_{33} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \quad : \textit{Tensor form}$$

- Dielectric response

$$\begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \quad : D_i = \varepsilon_{ij} E_j \; (tensor \;\; form)$$

Diagonalization : vanishing off-axis components

$$\begin{pmatrix} D_{x'} \\ D_{y'} \\ D_{z'} \end{pmatrix} = \begin{pmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{pmatrix} \begin{pmatrix} E_{x'} \\ E_{y'} \\ E_{z'} \end{pmatrix}$$

Energy density of the stored electric firld

$$Ue = \frac{1}{2} \vec{E} \cdot \vec{D} = \frac{1}{2} E_i \varepsilon_{ij} E_j$$

$x', \; y', \; z' : \; principal \;\; axis$

- Plane-wave propagation in anisotropic media

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \rightarrow (-j\vec{k})\times\vec{E} = -(j\omega)\vec{B}$$

$$\vec{k}\times\vec{E} = \omega\mu\vec{B}$$

$$\vec{\nabla}\times\vec{H} = \frac{\partial\vec{D}}{\partial t} \rightarrow (-j\vec{k})\times\vec{H} = (j\omega)\vec{D}$$

$$\vec{k}\times\vec{H} = -\omega\varepsilon\vec{D}$$

$$\vec{E} = \vec{E}(r,t)e^{j(\omega t-\vec{k}\cdot\vec{r})},\ \vec{H} = \vec{H}(r,t)e^{j(\omega t-\vec{k}\cdot\vec{r})}$$

- Wave equation

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} \rightarrow \vec{k} \times \vec{k} \times \vec{E} = -\omega^2 \mu\varepsilon \vec{E}$$

$$\therefore \vec{k} \times \vec{k} \times \vec{E} + \omega^2 \mu\varepsilon \vec{E} = 0$$

$$\varepsilon = \begin{pmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{pmatrix} : \textit{in the principal coordinate}$$

- This wave equation can be written

$$\vec{k}\times\vec{k}\times\vec{E}+\omega^2\mu\varepsilon\vec{E}=0$$

$$\begin{pmatrix} \omega^2\mu\varepsilon_x-k_y{}^2-k_z{}^2 & k_xk_y & k_xk_z \\ k_yk_x & \omega^2\mu\varepsilon_y-k_x{}^2-k_z{}^2 & k_yk_z \\ k_zk_x & k_zk_y & \omega^2\mu\varepsilon_z-k_x{}^2-k_y{}^2 \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}=0$$

$$\det\begin{vmatrix} \omega^2\mu\varepsilon_x-k_y{}^2-k_z{}^2 & k_xk_y & k_xk_z \\ k_yk_x & \omega^2\mu\varepsilon_y-k_x{}^2-k_z{}^2 & k_yk_z \\ k_zk_x & k_zk_y & \omega^2\mu\varepsilon_z-k_x{}^2-k_y{}^2 \end{vmatrix}=0$$

In terms of $\boldsymbol{k_z}$

$$k_z{}^2(\varepsilon_x k_x{}^2+\varepsilon_y k_y{}^2+\varepsilon_z k_z{}^2)-\omega^2\mu\varepsilon_x\varepsilon_y(k_x{}^2+k_y{}^2)$$
$$-\omega^2\mu\varepsilon_y\varepsilon_z(k_y{}^2+k_z{}^2)-\omega^2\mu\varepsilon_x\varepsilon_z(k_x{}^2+k_z{}^2)+(\omega^2\mu)^2\varepsilon_x\varepsilon_y\varepsilon_z=0$$

Therefore,

$$Ak_z{}^4+Bk_z{}^2+C=0 \;\rightarrow\; k_z : \pm k_{z1}, \pm k_{z2}$$

→ four points solutions

– k_x - k_y plane

$$\det\begin{vmatrix} \omega^2\mu\varepsilon_x - k_y{}^2 & k_x k_y & 0 \\ k_y k_x & \omega^2\mu\varepsilon_y - k_x{}^2 & 0 \\ 0 & 0 & \omega^2\mu\varepsilon_z - k_x{}^2 - k_y{}^2 \end{vmatrix} = 0$$

$$(\omega^2\mu\varepsilon_z - k_x{}^2 - k_y{}^2)[(\omega^2\mu\varepsilon_x - k_y{}^2)(\omega^2\mu\varepsilon_y - k_x{}^2) - k_x{}^2 k_y{}^2)] = 0$$

Solutions

$$1.\ \omega^2\mu\varepsilon_z = k_x{}^2 - k_y{}^2 \ \rightarrow k_x{}^2 + k_y{}^2 = \frac{\omega^2 n_z{}^2}{C^2} : circle$$

$$2.\ \omega^2\mu\varepsilon_x\varepsilon_y = \varepsilon_x k_x{}^2 + \varepsilon_y k_y{}^2 \rightarrow \frac{k_x{}^2}{\frac{\omega^2 n_y{}^2}{C^2}} + \frac{k_y{}^2}{\frac{\omega^2 n_x{}^2}{C^2}} : elliptic\ curve$$

– k_y - k_z plane

1. $k_y{}^2 + k_z{}^2 = \dfrac{\omega^2 n_x{}^2}{C^2}$: *circle*

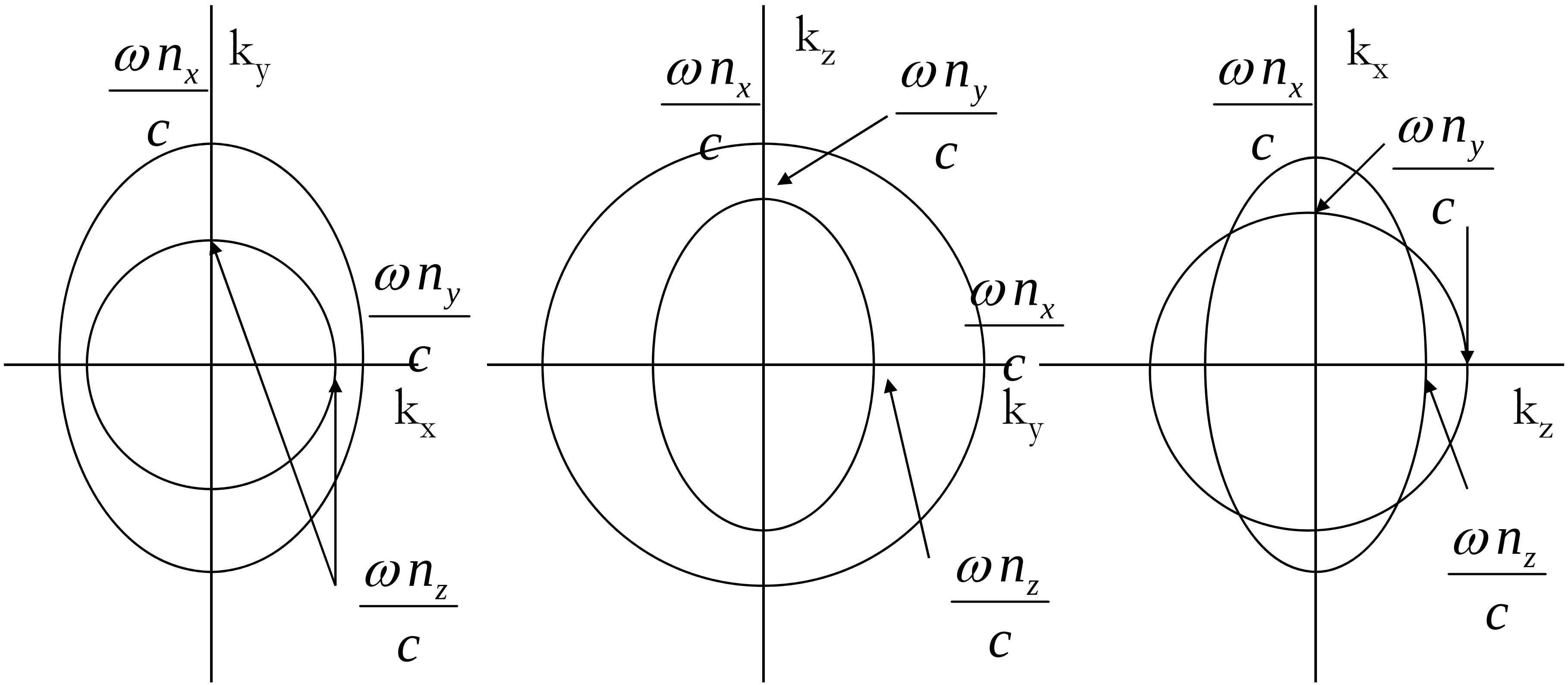
2. $\omega^2 \mu \varepsilon_y \varepsilon_z = \varepsilon_y k_y{}^2 + \varepsilon_z k_z{}^2 \rightarrow \dfrac{k_y{}^2}{\dfrac{\omega^2 n_z{}^2}{C^2}} + \dfrac{k_z{}^2}{\dfrac{\omega^2 n_y{}^2}{C^2}}$: *elliptic curve*

– k_z - k_x plane

1. $k_z{}^2 + k_x{}^2 = \dfrac{\omega^2 n_y{}^2}{C^2}$: *circle*

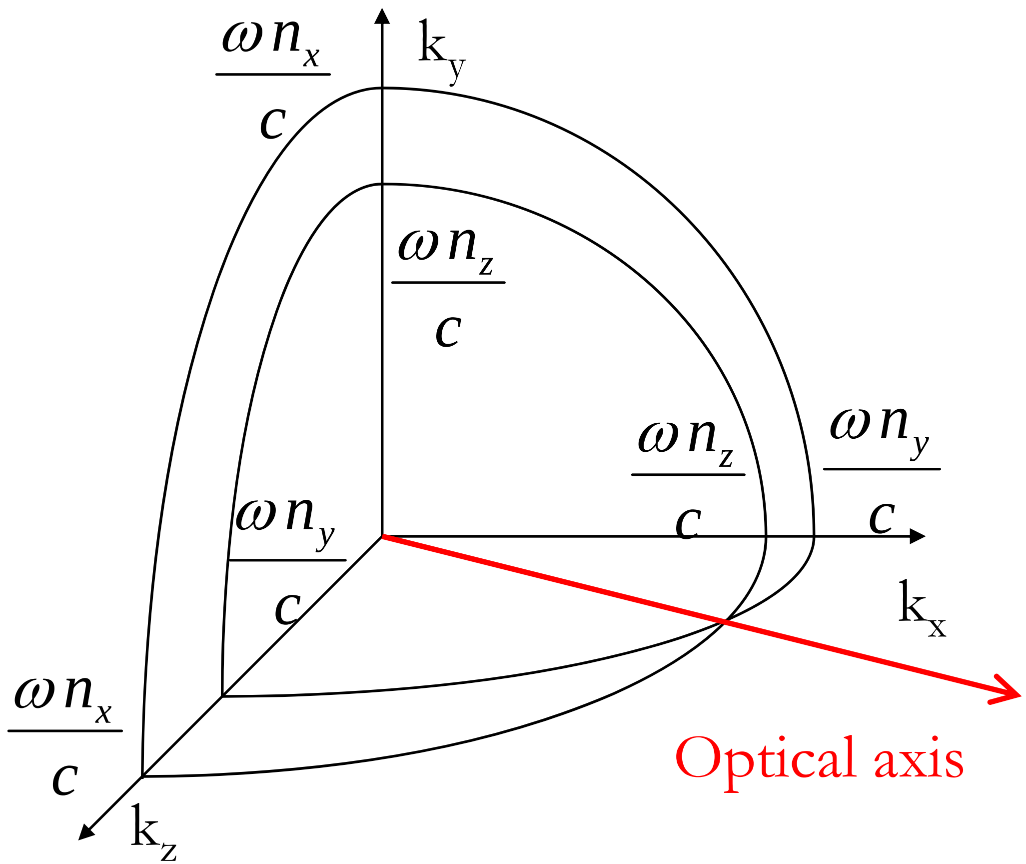
2. $\omega^2 \mu \varepsilon_z \varepsilon_x = \varepsilon_z k_z{}^2 + \varepsilon_x k_x{}^2 \rightarrow \dfrac{k_z{}^2}{\dfrac{\omega^2 n_x{}^2}{C^2}} + \dfrac{k_x{}^2}{\dfrac{\omega^2 n_z{}^2}{C^2}}$: *elliptic curve*

If $n_x > n_y > n_z$



$\frac{\omega n_x}{c}$
k_y
$\frac{\omega n_y}{c}$
k_x
$\frac{\omega n_z}{c}$
$\frac{\omega n_x}{c}$
k_z
$\frac{\omega n_y}{c}$
$\frac{\omega n_x}{c}$
k_y
$\frac{\omega n_z}{c}$
$\frac{\omega n_x}{c}$
k_x
$\frac{\omega n_y}{c}$
k_z
$\frac{\omega n_z}{c}$

If $\boldsymbol{k}$ surface



– Electric field direction

From $\det(k) = 0$

$$\det\begin{vmatrix} k_y(\omega^2\mu\varepsilon_x - k^2) & k_x(k^2 - \omega^2\mu\varepsilon_y) & 0 \\ k_y k_x & \omega^2\mu\varepsilon_y - k_x{}^2 - k_z{}^2 & k_y k_z \\ k_z(k^2 - \omega^2\mu\varepsilon_x) & 0 & k_x(\omega^2\mu\varepsilon_z - k^2) \end{vmatrix} = 0$$

$\therefore$

$$k_y(\omega^2\mu\varepsilon_x - k^2)E_x + k_x(k^2 - \omega^2\mu\varepsilon_y)E_y = 0$$

$$E_y = -\frac{k_y}{k_x}\frac{\omega^2\mu\varepsilon_x - k^2}{k^2 - \omega^2\mu\varepsilon_y}E_x$$

$$k_z(k^2 - \omega^2\mu\varepsilon_x)E_x + k_x(\omega^2\mu\varepsilon_z - k^2)E_z = 0$$

$$E_z = -\frac{k_z}{k_x}\frac{k^2 - \omega^2\mu\varepsilon_x}{\omega^2\mu\varepsilon_z - k^2}E_x$$

Electric field E components are

$$\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = E_x \frac{k^2 - \omega^2 \mu \varepsilon_x}{k_x} \begin{pmatrix} \dfrac{k_x}{k^2 - \omega^2 \mu \varepsilon_x} \\ \dfrac{k_y}{k^2 - \omega^2 \mu \varepsilon_y} \\ \dfrac{k_z}{k^2 - \omega^2 \mu \varepsilon_z} \end{pmatrix}$$

Thus,

$$\begin{pmatrix} \dfrac{k_x}{k^2 - \omega^2 \mu \varepsilon_x} \\ \dfrac{k_y}{k^2 - \omega^2 \mu \varepsilon_y} \\ \dfrac{k_z}{k^2 - \omega^2 \mu \varepsilon_z} \end{pmatrix}$$

-.Electric field is related with propagation

-.Two set of ***k*** value : two polarizations

k_1, k_2

-.Electric field E components related to refractive index

$$\det\begin{vmatrix} k_y(\omega^2\mu\varepsilon_x - k^2) & k_x(k^2 - \omega^2\mu\varepsilon_y) & 0 \\ k_y k_x & \omega^2\mu\varepsilon_y - k_x{}^2 - k_z{}^2 & k_y k_z \\ k_z(k^2 - \omega^2\mu\varepsilon_x) & 0 & k_x(\omega^2\mu\varepsilon_z - k^2) \end{vmatrix} = 0$$

From here,

$$k_x k_y(\omega^2\mu\varepsilon_x - k^2)(\omega^2\mu\varepsilon_y - k_x{}^2 k_z{}^2)(\omega^2\mu\varepsilon_z - k^2) +$$
$$k_x k_y k_z{}^2(k^2 - \omega^2\mu\varepsilon_y)(k^2 - \omega^2\mu\varepsilon_x) + k_x{}^3 k_y(k^2 - \omega^2\mu\varepsilon_y)(k^2 - \omega^2\mu\varepsilon_z) = 0$$

$$\therefore \quad \frac{k_x{}^2}{k^2 - \omega^2\mu\varepsilon_x} + \frac{k_y{}^2}{k^2 - \omega^2\mu\varepsilon_y} + \frac{k_z{}^2}{k^2 - \omega^2\mu\varepsilon_z} = 0$$

$$\vec{k} = \frac{\omega n}{c}\vec{s}, \quad \mu_0 \varepsilon_0 = \frac{1}{c^2}$$

$$\therefore \quad \frac{n^2 s_x{}^2}{n^2 - \varepsilon_x/\varepsilon_0} + \frac{n^2 s_y{}^2}{n^2 - \varepsilon_y/\varepsilon_0} + \frac{n^2 s_z{}^2}{n^2 - \varepsilon_z/\varepsilon_0} = 0$$

Therefore, electric field ***E*** components are

$$\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \Rightarrow \begin{pmatrix} \dfrac{k_x}{k^2 - \omega^2 \mu \varepsilon_x} \\ \dfrac{k_y}{k^2 - \omega^2 \mu \varepsilon_y} \\ \dfrac{k_z}{k^2 - \omega^2 \mu \varepsilon_z} \end{pmatrix} \Rightarrow \frac{cn}{\omega} \begin{pmatrix} \dfrac{s_x}{n^2 - \varepsilon_x/\varepsilon_0} \\ \dfrac{s_y}{n^2 - \varepsilon_y/\varepsilon_0} \\ \dfrac{s_z}{n^2 - \varepsilon_z/\varepsilon_0} \end{pmatrix}$$

Two polarization because of two refractive index n_1, n_2

– Vector field investigation

$$\vec{\nabla}\cdot\vec{D}_1 = 0 \rightarrow (-j\vec{k}_1)\cdot\vec{D}_1 = 0 \rightarrow \vec{k}_1 \perp \vec{D}_1$$

$$\vec{\nabla}\cdot\vec{D}_2 = 0 \rightarrow (-j\vec{k}_2)\cdot\vec{D}_2 = 0 \rightarrow \vec{k}_2 \perp \vec{D}_2$$

From above equstion $\vec{D}_1\cdot\vec{D}_2 = 0,\ and\ \vec{s} // \vec{k}_1 // \vec{k}_2$

$$\therefore \vec{D}_1 \perp \vec{D}_2 \perp \vec{s} = 0$$

- Orthogonality properties

$$\vec{\nabla}\times\vec{H}=\frac{\partial\vec{D}}{\partial t}\rightarrow\vec{D}=-\frac{n}{c}\vec{s}\times\vec{H}\quad\Rightarrow\quad\vec{D}\perp\vec{s},\ \vec{D}\perp\vec{H},\ \vec{H}\perp\vec{s}$$

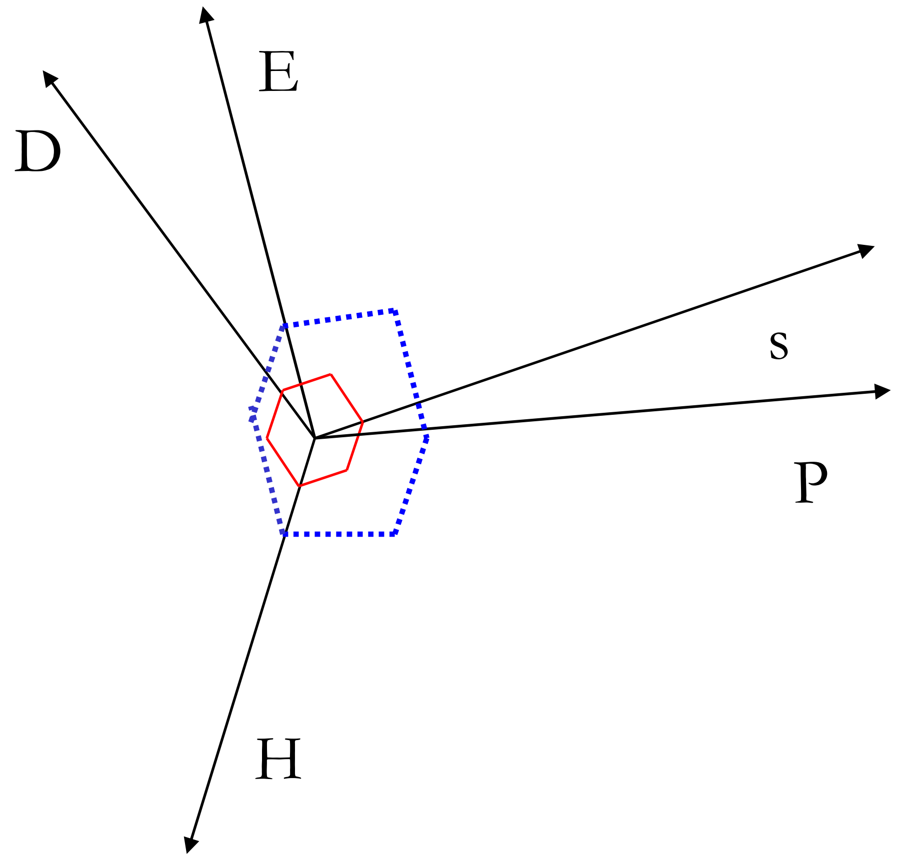
$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}\rightarrow\vec{H}=-\frac{n}{\mu c}\vec{s}\times\vec{E}\quad\Rightarrow\quad\vec{H}\perp\vec{s},\vec{H}\perp\vec{E}$$

$$\vec{\nabla}\cdot\vec{B}=0\rightarrow\omega\frac{n}{c}\vec{s}\times\vec{B}\quad\Rightarrow\quad\vec{B}\perp\vec{s}$$

$$\vec{\nabla}\cdot\vec{D}=0\rightarrow\omega\frac{n}{c}\vec{s}\times\vec{D}\quad\Rightarrow\quad\vec{D}\perp\vec{s}$$

There is no proof that $\boldsymbol{E}//\boldsymbol{D}$ and $\boldsymbol{P}//\boldsymbol{s}$
If $\boldsymbol{B}//\boldsymbol{H}$, then $\boldsymbol{H}$ and $\boldsymbol{s}$ is orthogonal

Vector orthogonality



$$\vec{D} \perp \vec{s},\ \vec{D} \perp \vec{H},\ \vec{H} \perp \vec{s}$$

$$\vec{H} \perp \vec{s}, \vec{H} \perp \vec{E}$$

$$\vec{B} \perp \vec{s}$$

$$\vec{D} \perp \vec{s}$$

$$\vec{E} \times \vec{H} = \vec{P}$$

• The index ellipsoid

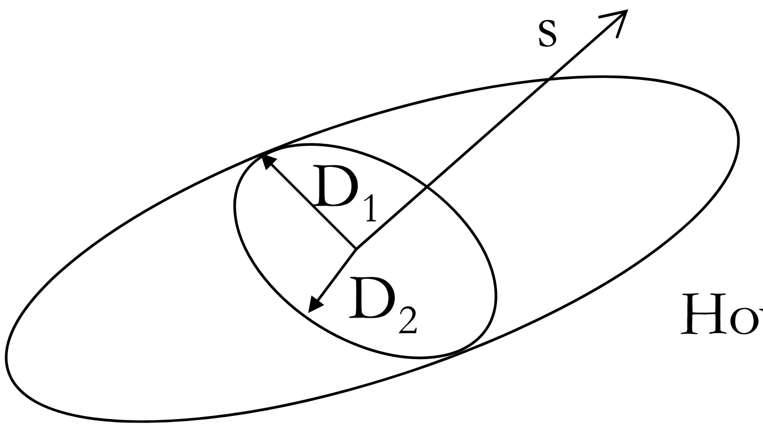
Energy density of stored electric field in anisotropic material

$$U_e = \frac{1}{2}\vec{D}\cdot\vec{E} = \frac{1}{2}E_i\varepsilon_{ij}E_j \rightarrow \frac{D_x^{\,2}}{\varepsilon_x} + \frac{D_y^{\,2}}{\varepsilon_y} + \frac{D_z^{\,2}}{\varepsilon_z} = 2U_e$$

$$let\ \frac{D_i}{\sqrt{2U_e\varepsilon_0}} = k,\ \ k = x, y, z \qquad \frac{1}{2U_e\varepsilon_0}\left(\frac{D_x^{\,2}}{\varepsilon_x} + \frac{D_y^{\,2}}{\varepsilon_y} + \frac{D_z^{\,2}}{\varepsilon_z}\right) = 1$$

$$\frac{x^2}{n_x} + \frac{y^2}{n_y} + \frac{z^2}{n_z} = 1$$:the index ellipsoid or the optical indicatrix

Method of index ellipsoid



How to determine the ***n_1, n_2, D_1, D_2***?

We define the impermeability η_{ij} $\quad \eta_{ij} = \varepsilon_0(\varepsilon_r^{-1})_{ij}$

Wave equation regarding ***s***

$$\vec{k}\times\vec{k}\times\vec{E}+\omega^2\mu\varepsilon\vec{E}=0 \rightarrow \vec{s}\times(\vec{s}\times\eta\vec{D})+\frac{\vec{D}}{n^2}=0$$

$$\uparrow$$

$$\vec{k}=\frac{n\omega}{c}\vec{s}$$

Assuming ***s*** has one direction $s = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $\vec{s} \cdot \vec{D} = 0 \rightarrow D = \begin{pmatrix} Dx \\ Dy \\ 0 \end{pmatrix}$

Therefore,

$$\begin{pmatrix} \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{21} & \eta_{22} & \eta_{23} \\ 0 & 0 & 0 \end{pmatrix} \vec{D} = \frac{1}{n^2} \vec{D}$$

ignore (third column, since $D_z = 0$)

because $\vec{s} \cdot \vec{D} = 0$ (third row)

$$\therefore \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix} \vec{D} = \frac{1}{n^2} \begin{pmatrix} \overline{D}_1 \\ \overline{D}_2 \end{pmatrix} \rightarrow (\eta_t - \frac{1}{n^2}) \vec{D} = 0$$

η_t :transverse impermeability tensor

→ the polarization vectors of the normal modes are eigenvectors of the transverse impermeability tensor with eigenvalues $1/n^2$

→ 2×2 symmetric tensor : orthogonal eigenvector $(n_1, D_1 : n_2, D_2)$

- Phase velocity, group velocity, and energy velocity

$$\vec{v}_p = \frac{\omega}{k}\vec{s} \qquad : \ phase velocity$$

$$\vec{v}_g = \vec{\nabla}_k \omega(k) \qquad : \ group velocity$$

$$\vec{v}_e = \frac{\vec{s}}{U} \qquad : \ energy velocity$$

s : poynting vector
U: energy density

In an isotropic material, $\vec{v}_g = \vec{v}_e$

• Classification of anisotropic media

1) Biaxial : two optical axis, $n_x \neq n_y \neq n_z$
2) Uniaxial : one optical axis
   two principal indices are the same.

$$Let,\ n_0^{\ 2} = \frac{\varepsilon_x}{\varepsilon_0} = \frac{\varepsilon_y}{\varepsilon_0}, \qquad n_e^{\ 2} = \frac{\varepsilon_z}{\varepsilon_0}$$

n_o: ordinary refractive index
n_e: extraordinary refractive index

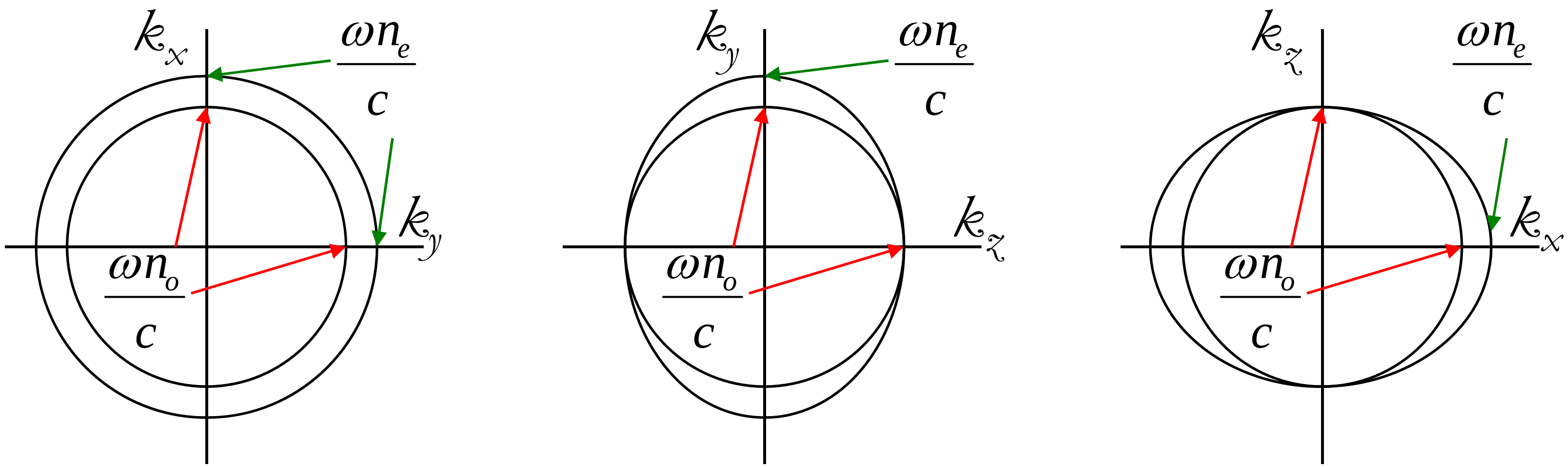
$\det(k) = 0$ normal ***k*** plane

$$\left(\frac{k^2}{n_o^{\ 2}} - \frac{\omega^2}{c^2}\right)\left(\frac{k_x^{\ 2} + k_y^{\ 2}}{n_e^{\ 2}} + \frac{k_z^{\ 2}}{n_o^{\ 2}} - \frac{\omega^2}{c^2}\right) = 0$$

$$\therefore \frac{k_x^{\ 2}+k_y^{\ 2}+k_z^{\ 2}}{n_o^{\ 2}}=\frac{\omega^2}{c^2}, \quad \frac{k_x^{\ 2}}{n_e^{\ 2}}+\frac{k_y^{\ 2}}{n_e^{\ 2}}+\frac{k_z^{\ 2}}{n_o^{\ 2}}=\frac{\omega^2}{c^2}$$

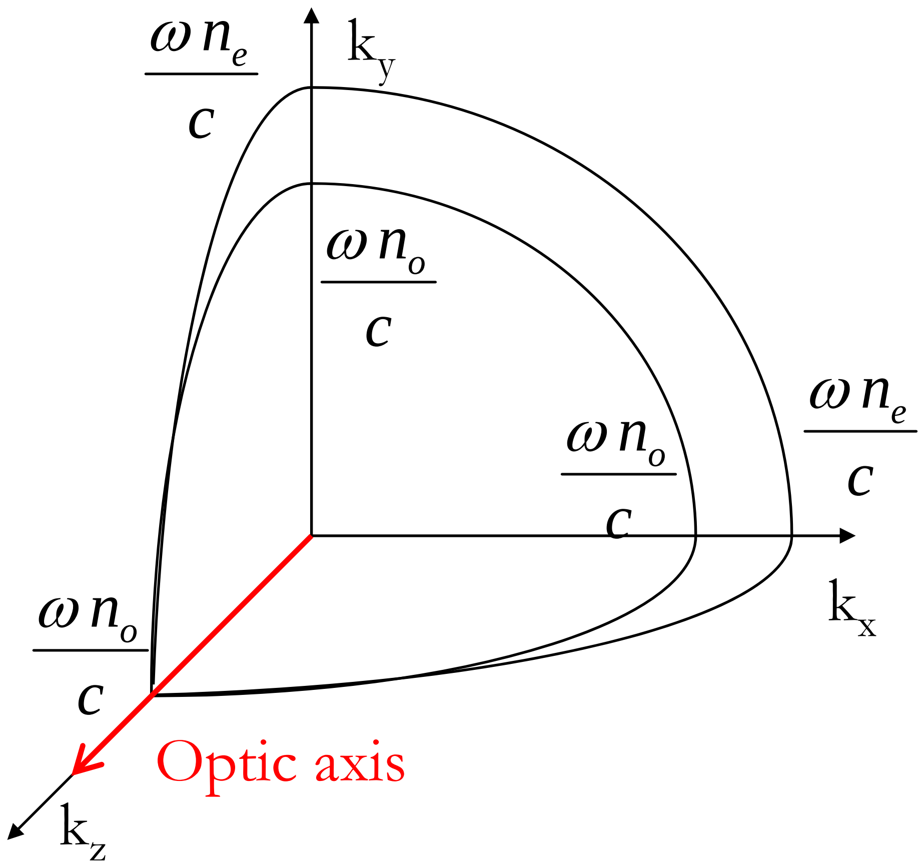
$$\therefore \frac{k_x^{\ 2}+k_y^{\ 2}+k_z^{\ 2}}{(\omega^2 n_o^{\ 2}/c^2)}=1, \quad \frac{k_x^{\ 2}}{(\omega^2 n_e^{\ 2}/c^2)}+\frac{k_y^{\ 2}}{(\omega^2 n_e^{\ 2}/c^2)}+\frac{k_z^{\ 2}}{(\omega^2 n_o^{\ 2}/c^2)}=1$$

If $\varepsilon_x=\varepsilon_y<\varepsilon_z$ ($n_o < n_e$)



k_x-k_y plane $\quad$ k_y-k_z plane $\quad$ k_z-k_x plane

The normal *k* surface



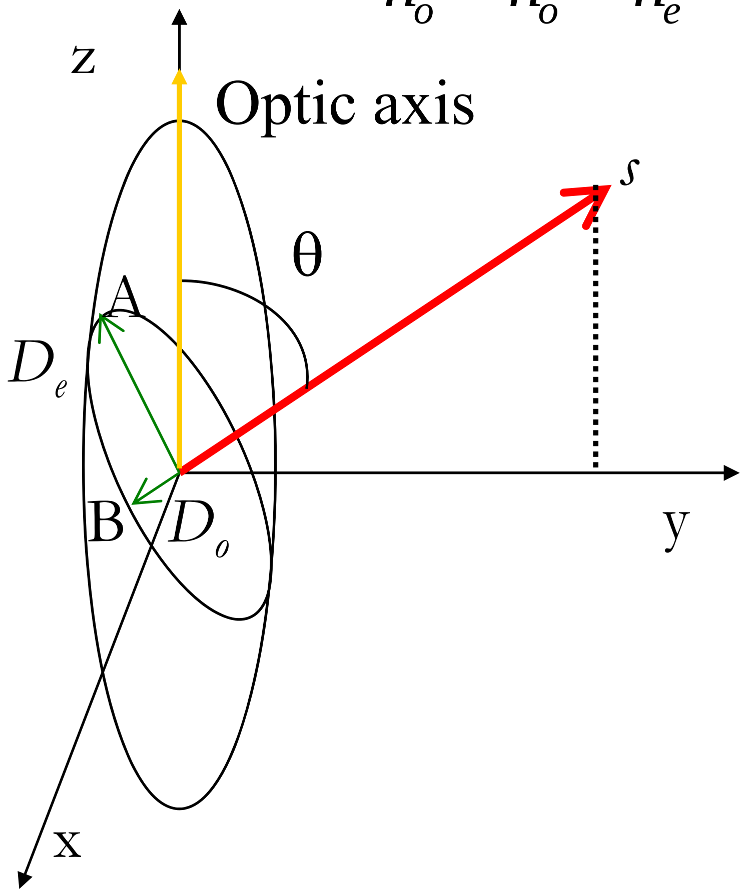
$n_o < n_e$: positive uniaxial crystal

$n_o > n_e$: negative uniaxial crystal

$n_o = n_e$ or $n_x = n_y = n_z$: isotropic material

• Light propagation in unaxial media

Index ellipsoid $$\frac{x^2}{n_o} + \frac{y^2}{n_o} + \frac{z^2}{n_e} = 1 : optic\,axis\,is\ z\ axis$$



Positive material

Polarization of ordinary displacement vector d_o

$$\vec{d_o} = \frac{\vec{k} \times \vec{a_z}}{\left|\vec{k} \times \vec{a_z}\right|}$$

Polarization of extraordinary displacement vector d_o

$$\vec{d_e} = \frac{\vec{k} \times \vec{d_o}}{\left|\vec{k} \times \vec{d_o}\right|}$$

E-ray and O-ray

As propagation direction *s* is changed

θ is changed ⟶ n_o fixed, D_o fixed
n_e changed, D_e changed

$$n_e(\theta) = \begin{cases} n_o \ at\ \theta = 0 \\ n_e \ at\ \theta = \pi/2 \end{cases}$$

In order to achieve $n_e(\theta)$, using normal surface

$$\left(\frac{k^2}{n_o^2} - \frac{\omega^2}{c^2}\right)\left(\frac{k_x^2 + k_y^2}{n_e^2} + \frac{k_z^2}{n_o^2} - \frac{\omega^2}{c^2}\right) = 0$$

On the y-z plane :

$$k_x = 0, \quad k_z = k\cos\theta = \frac{n\omega}{c}, \quad k_y{}^2 = k^2 - k_z{}^2 = (\frac{n\omega}{c})^2 \sin^2\theta$$

then, for E-wave

$$\frac{(\frac{n\omega}{c})^2 \sin^2\theta}{n_e{}^2} + \frac{(\frac{n\omega}{c})^2 \cos^2\theta}{n_o{}^2} = \frac{\omega^2}{c^2}$$

therefore,

$$\frac{\sin^2\theta}{n_e{}^2} + \frac{\cos^2\theta}{n_o{}^2} = \frac{1}{n_e{}^2(\theta)}$$

:E-ray

then, for O-wave

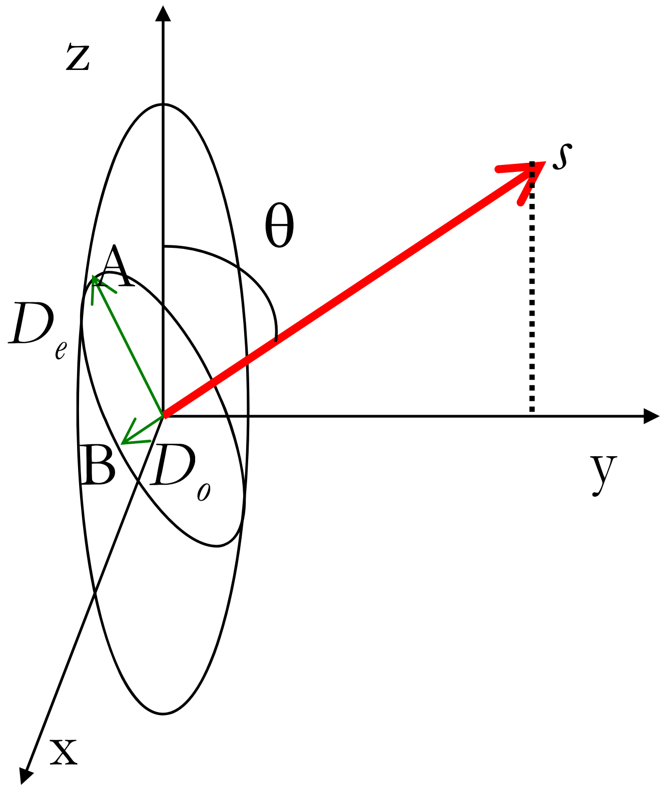
$$\frac{\left(\frac{n\omega}{c}\right)^2 \sin^2\theta + \left(\frac{n\omega}{c}\right)^2 \cos^2\theta}{n_o^{\,2}} = \frac{\omega^2}{c^2}$$

therefore,

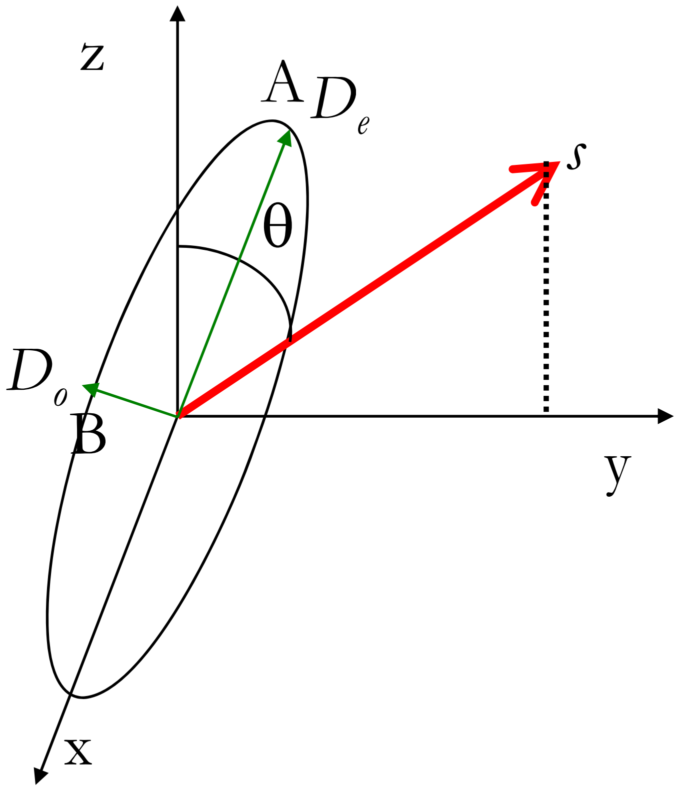
$$\frac{n_o^{\,2}(\theta)}{n_o^{\,2}} = 1 \quad \rightarrow n_o^{\,2}(\theta) = n_o^{\,2}$$

:o-ray

Retardation btn. homeotropic and homogeneous LC layer



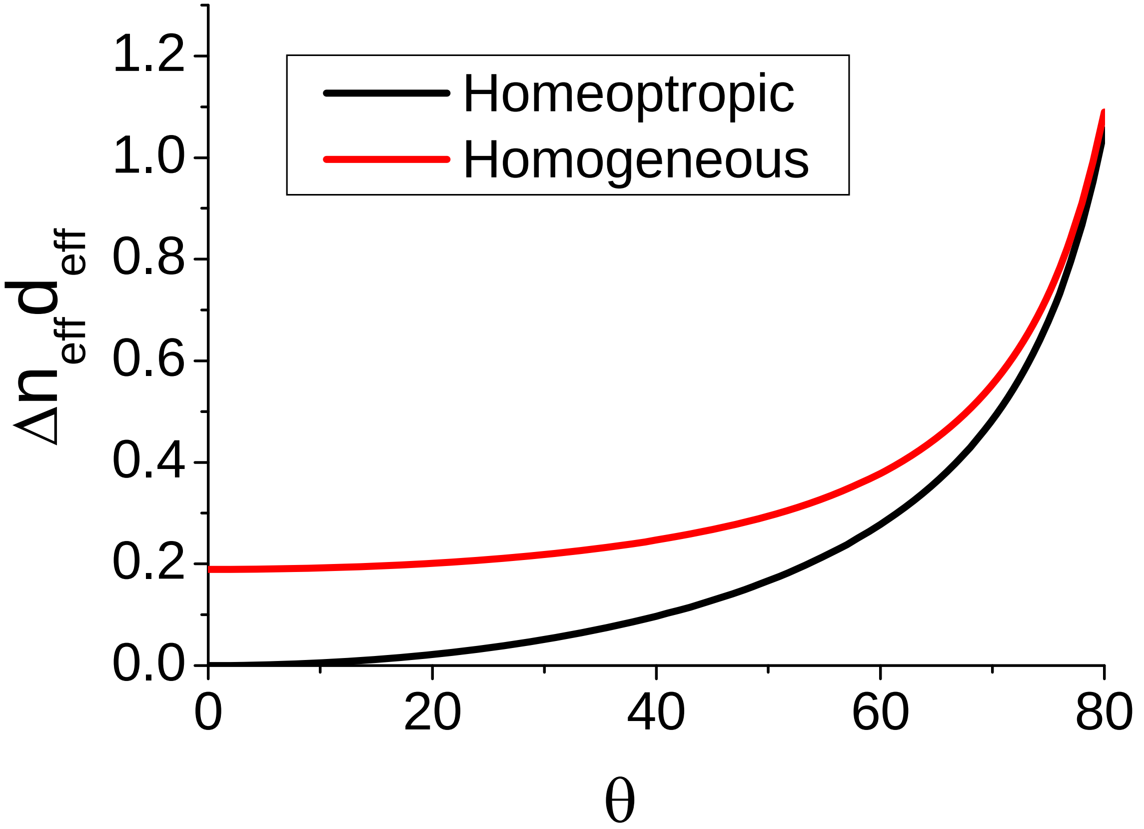
homogeneous LC layer



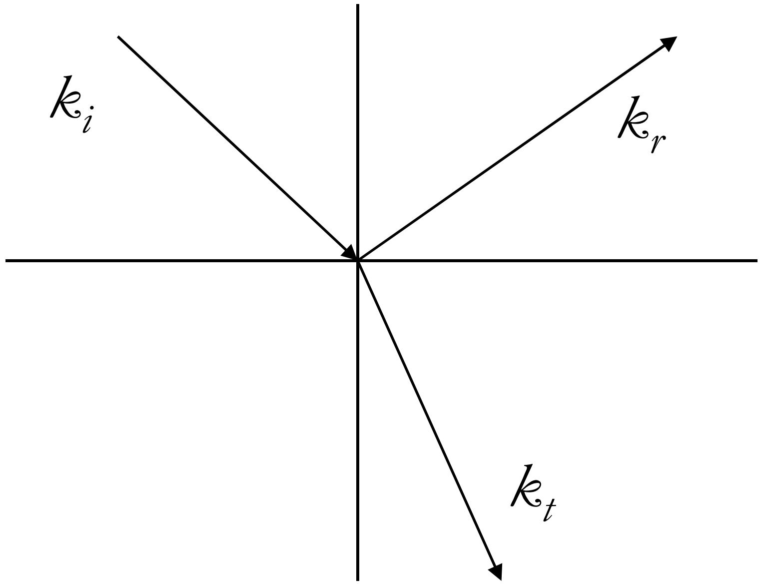
homeotropic LC layer

$$\frac{1}{n_e^{\,2}(\theta)} = \frac{\sin^2\theta}{n_e^{\,2}} + \frac{\cos^2\theta}{n_o^{\,2}}, \quad n_o^{\,2}(\theta) = n_o^{\,2}$$

$$\therefore \Delta n_{eff} = n_e(\theta) - n_o, \quad d_{eff} = \frac{d}{\cos\theta}$$

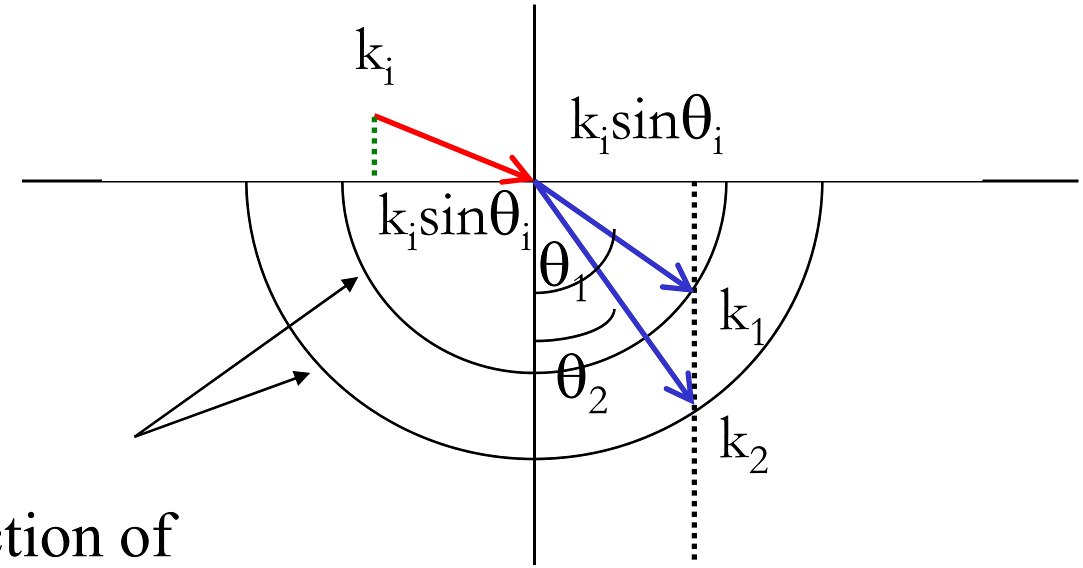


- Double refraction at a boundary



All reflected and refracted rays are same tagential components of the wave number, even, at a anisotropic boundary,
So

Double refraction at a anisotropic media boundary



Intersection of normal surface with plane of incidence

$$k_i \sin\theta_i = k_1 \sin\theta_1 = k_2 \sin\theta_2$$

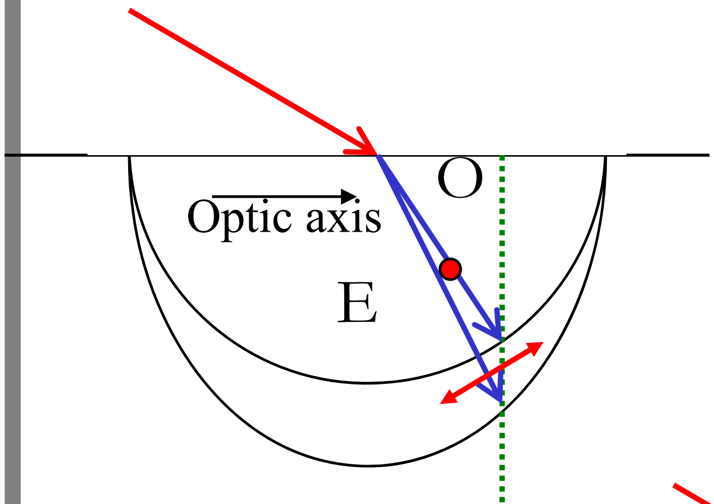
Snell's law ? $k_1, k_2 \rightarrow k_1(\theta_1) \quad k_2(\theta_2)$

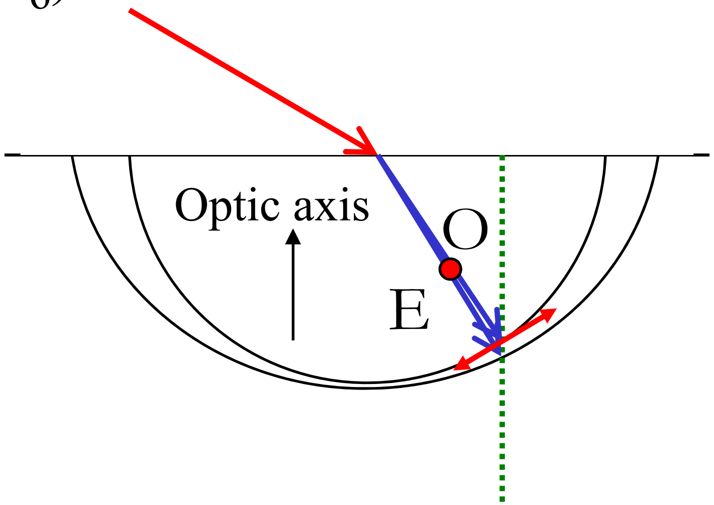
k_1, k_2 are dependent on their refracted angle

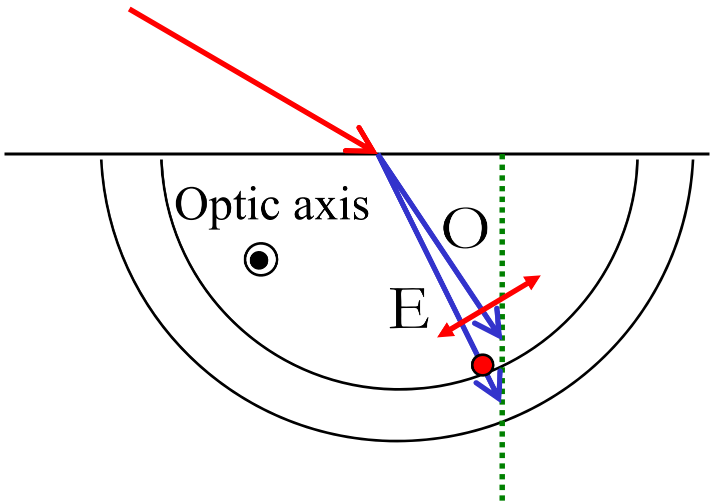
So, graphic method

Graphic method of double refraction on a uniaxial media

Positive material($n_e > n_o$)

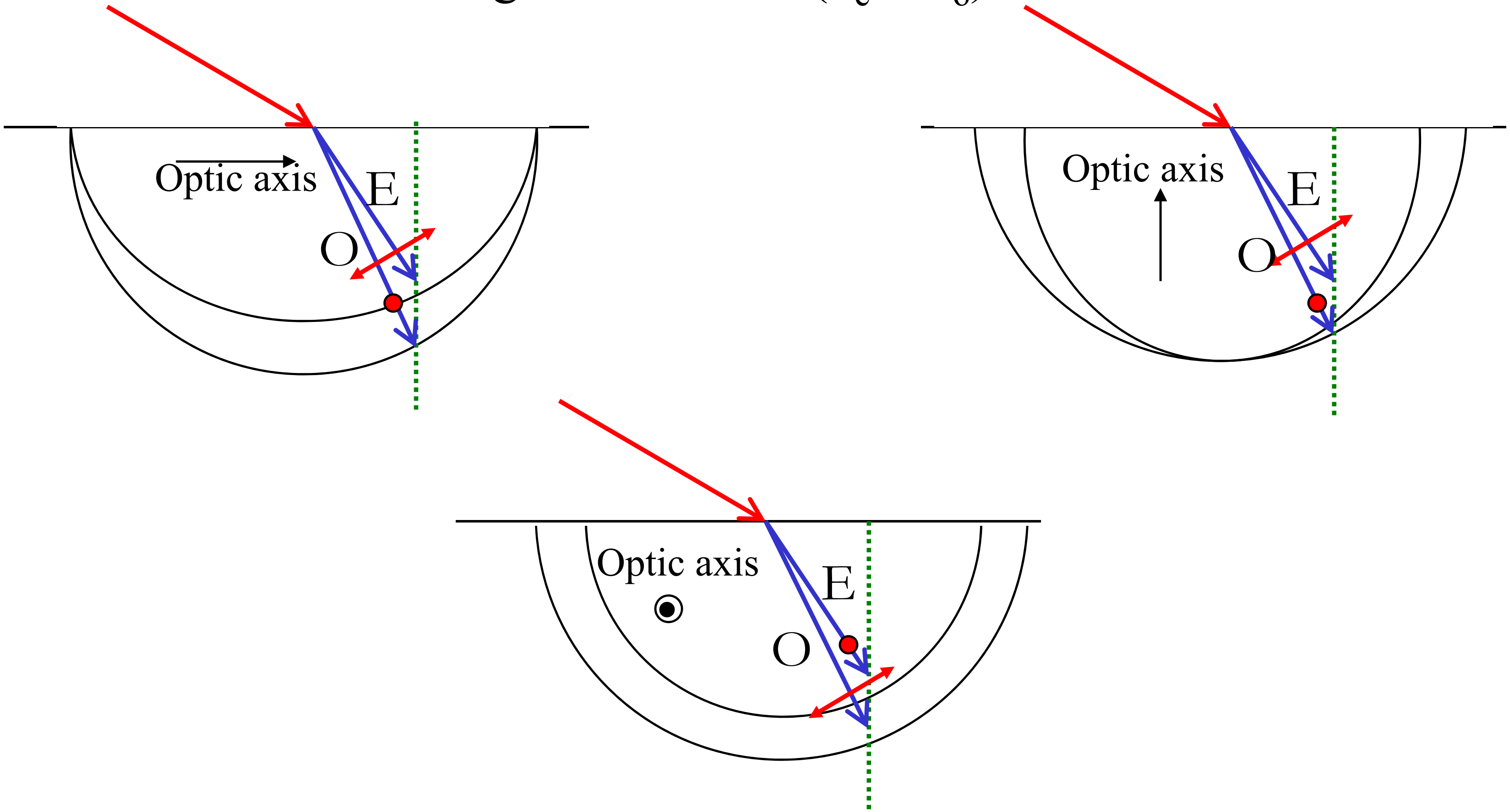






Graphic method of double refraction on a uniaxial media

Negative material($n_e < n_o$)



Reference

- Amnon Yariv, Pochi Yeh, “Optical waves in crystals”, wileys